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Description of regular and intra-regular ordered semigroups by tripolar fuzzy ideals

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Abstract

A mathematical notion dealing with tripolar information is called tripolar fuzzy sets. It was first used to study the properties of Γ-semigroups. In 2022, ordered semigroups were the first considered by tripolar fuzzy sets. It was established that in regular and intra-regular ordered semigroups, the concepts of tripolar fuzzy interior ideals and tripolar ideals are equivalent. The primary goal of this study is to employ tripolar fuzzy left and right ideals to characterize regular and intra-regular ordered semigroups. Additionally, a connection between tripolar fuzzy left (resp., right) ideals and left (resp., right) ideals in ordered semigroups is given.

Keywords: Ordered semigroup, tripolar fuzzy left ideal, tripolar fuzzy right ideal, regularity.

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1. Introduction

The idea of fuzzy sets is a mathematical concept that deals with uncertainty in real-world problems. Zadeh [10] invented fuzzy sets for the first time in 1965. Fuzzy sets can be used to study topics in computer science, artificial intelligence, control engineering, robotics, and automata theory. Although fuzzy sets can be used in many scientific investigations, they are limited to several decision-related issues. Mathematical objects called intuitionistic and bipolar fuzzy sets take care of issues that fuzzy sets cannot handle in some circumstances. Atanassov [1] introduced fuzzy sets with intuitive properties. This idea can display both the degree and non-degree of memberships, which helps with ambiguity. Therefore, it can be used to study decision-making problems. Another notion that can be dealt with decision-making difficulties is bipolar fuzzy sets. In 1994, Zhang [11] discussed this concept. There was a discussion of the differences between intuitionistic and bipolar fuzzy sets in 2001 (see [6]).

As a generalization of intuitionistic and bipolar fuzzy sets, Rao [7] presented the idea of tripolar fuzzy sets. Intuitionistic and bipolar fuzzy sets are combined to create this concept. One can use tripolar fuzzy

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sets to examine a scenario with tripolar information. When tripolar fuzzy sets were initially introduced, this notion was used for the first time to explore the structural characteristics of Γ-semigroups (see [7]). Later, tripolar fuzzy sets were used in [8] to evaluate the properties of Γ-semirings. The current authors used tripolar fuzzy sets to examine ideals in ordered semigroups. Wattanasiripong et al. [9] presented many types of tripolar fuzzy ideals in ordered semigroups. They demonstrated that in regular and intra-regular ordered semigroups, tripolar fuzzy interior and tripolar (two-sided) ideals coincide.

By the above motivation, we aim to characterize regular and intra-regular ordered semigroups in which any tripolar fuzzy interior and (two-sided) ideal are the same. Before that, we discuss the tripolar fuzzy left and right ideals: the tools used to characterize such classes of ordered semigroups. Additionally, we represent left and right ideals in ordered semigroups using tripolar fuzzy left and right ideals. The paper is organized as follows. Section 2 provides all the necessary basic knowledge used in this paper. Moreover, Some known results are also presented. In our primary section, Section 3, we present the main objectives of our paper. We summarize our detail in Section 4.

2. Preliminaries

This section recalls some basic terminologies of ordered semigroups and tripolar fuzzy sets, which will be used in the paper.

The algebraic system $\langle S; \cdot, \leqslant \rangle$ is said to be an *ordered semigroup* if $\langle S; \cdot \rangle$ is a semigroup and $\langle S; \leqslant \rangle$ is a partially ordered set such that $a \cdot c \leqslant b \cdot c$ and $c \cdot a \leqslant c \cdot b$ whenever $a \leqslant b$ for any $a, b, c \in S$. We denote an ordered semigroup $\langle S; \cdot, \leqslant \rangle$ by the bold letter $\mathbf{S}$ of its universe set. Moreover, we denote the product $x \cdot y$ by xy.

Let $\mathbf{S}$ be an ordered semigroup and A, B, and C nonempty subsets of S. We define the sets AB and $(C]$ by $\{ab : a \in A \text{ and } b \in B\}$ and $\{t \in S : t \leqslant c \text{ for some } c \in C\}$, respectively. A nonempty subset A of S is called a *left* (*resp., right*) *ideal* [3] of $\mathbf{S}$ if:

(1) $SA \subseteq A$ (resp., $AS \subseteq A$);
(2) $(A] = A$, equivalently, $x \in A$ whenever $x \leqslant y$ and $y \in A$.

A set A is referred to be an (*two-sided*) *ideal* of $\mathbf{S}$ if it is both a left and a right ideal of $\mathbf{S}$.

Let $\mathbf{S}$ be an ordered semigroup. Then $\mathbf{S}$ is said to be:

(1) *regular* [3] if there exists $x \in S$ such that $a \leqslant axa$ for any $a \in S$;
(2) *intra-regular* [4] if there exist $x, y \in S$ such that $a \leqslant xa^2y$ for any $a \in S$.

Several algebraists intensively study the above classes of ordered semigroups. Additionally, these two classes of ordered semigroups can be studied using the previously introduced notions of left and right ideals.

Lemma 2.1 ([2, 5])**.** *Let* $\mathbf{S}$ *be an ordered semigroup. Then*

(1) $\mathbf{S}$ *is regular if and only if* $R \cap L = (RL]$ *for any left ideal* L *and right ideal* R *of* $\mathbf{S}$*;*
(2) $\mathbf{S}$ *is intra-regular if and only if* $L \cap R \subseteq (LR]$ *for any left ideal* L *and right ideal* R *of* $\mathbf{S}$*.*

Definition 2.2 ([7])**.** Let X be a nonempty set. A set f of the form

$$f := \{(x, f^+(x), f^*(x), f^-(x)) : x \in X \text{ and } 0 \leqslant f^+(x) + f^*(x) \leqslant 1\},$$

where $f^+ : X \to [0, 1]$, $f^* : X \to [0, 1]$, and $f^- : X \to [-1, 0]$, is called a *tripolar fuzzy set of* X. The membership degree $f^+(x)$ characterizes the extent to which the element x satisfies the property corresponding to tripolar fuzzy set f, $f^*(x)$ characterizes the extent that the element x satisfies the not property (irrelevant) corresponding to tripolar fuzzy set f, and $f^-(x)$ characterizes the extent that the element x satisfies the implicit counter property of tripolar fuzzy set f.

For simplicity, we write for any tripolar fuzzy set $\{(x, f^+(x), f^*(x), f^-(x)) : x \in X \text{ and } 0 \leqslant f^+(x) + f^*(x) \leqslant 1\}$ by the notation f. Therefore, the notation $f(x)$ means $(f^+(x), f^*(x), f^-(x))$ for any $x \in X$. Let $A \subseteq X$. A tripolar fuzzy set χ_A of X, where

$$\chi_A^+(x) := \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise}, \end{cases} \quad \chi_A^*(x) := \begin{cases} 0, & \text{if } x \in A, \\ 1, & \text{otherwise}, \end{cases} \quad \chi_A^-(x) := \begin{cases} -1, & \text{if } x \in A, \\ 0, & \text{otherwise}, \end{cases}$$

for any $x \in X$, is called *the characteristic tripolar fuzzy set of* A *in* X. In particular, if $A = X$, then we denote the tripolar fuzzy set χ_A of A in X by 1. On the other hand, if $A = \emptyset$, then we denote χ_A by 0. By the definition of tripolar fuzzy sets of X, we can regard:

(1) a fuzzy set $f\colon X \to [0,1]$ of X by a tripolar fuzzy set $(f, 1^*, 1^-)$;
(2) an intuitionistic fuzzy set $(f^+, f^*)\colon X \to [0,1] \times [0,1]$ of X by a tripolar fuzzy set $(f^+, f^*, 1^-)$;
(3) a bipolar fuzzy set $(f^+, f^-)\colon X \to [0,1] \times [-1,0]$ of X by a tripolar fuzzy set $(f^+, 1^*, f^-)$.

Let $u_1 = (x_1, y_1, z_1), u_2 = (x_2, y_2, z_2) \in [0,1] \times [0,1] \times [-1,0]$ such that $0 \leqslant x_1 + y_1 \leqslant 1$ and $0 \leqslant x_2 + y_2 \leqslant 1$. Then we denote by $u_1 \leqslant u_2$ if $x_1 \leqslant x_2$, $y_1 \geqslant y_2$, and $z_1 \geqslant z_2$. If $u_1 \leqslant u_2$ and $u_2 \leqslant u_1$, then we write $u_1 = u_1$. For any two tripolar fuzzy sets f and g of X, we designate by $f \subseteq g$ whenever $f(x) \leqslant g(x)$ for all $x \in X$. Moreover, we define a tripolar fuzzy set $f \cap g = ((f \cap g)^+, (f \cap g)^*, (f \cap g)^-)$ of X by

$$(f \cap g)^+(x) := \min\{f^+(x), g^+(x)\}, \quad (f \cap g)^*(x) := \max\{f^*(x), g^*(x)\}, \quad (f \cap g)^-(x) := \max\{f^-(x), g^-(x)\},$$

for all $x \in X$.

Remark 2.3. Let χ_A be the characteristic tripolar fuzzy set of A in X, and $x, y \in X$. Suppose that $\chi_A(y) \leqslant \chi_A(x)$. Then it is not difficult to illustrate that $x \in A$ whenever $y \in A$.

It is possible to examine the structure of ordered semigroups by using the concept of tripolar fuzzy sets. Let **S** be an ordered semigroup, and $a \in S$. We define $\mathbf{S}_a := \{(x, y) \in S \times S : a \leqslant xy\}$. For any two tripolar fuzzy sets f and g of **S**, we define a tripolar fuzzy set $f \circ g = ((f \circ g)^+, (f \circ g)^*, (f \circ g)^-)$ of S by

$$(f \circ g)^+(x) := \begin{cases} \bigvee_{(u,v) \in \mathbf{S}_x} \{\min\{f^+(u), g^+(v)\}\}, & \text{if } \mathbf{S}_x \neq \emptyset, \\ 0, & \text{otherwise}, \end{cases}$$

$$(f \circ g)^*(x) := \begin{cases} \bigwedge_{(u,v) \in \mathbf{S}_x} \{\max\{f^*(u), g^*(v)\}\}, & \text{if } \mathbf{S}_x \neq \emptyset, \\ 1, & \text{otherwise}, \end{cases}$$

and

$$(f \circ g)^-(x) := \begin{cases} \bigwedge_{(u,v) \in \mathbf{S}_x} \{\max\{f^-(u), g^-(v)\}\}, & \text{if } \mathbf{S}_x \neq \emptyset, \\ 0, & \text{otherwise}, \end{cases}$$

for all $x \in S$.

We define the notion of tripolar fuzzy ideals of ordered semigroups as follows.

Definition 2.4 ([9]). Let **S** be an ordered semigroup. A tripolar fuzzy set f of S is called a *tripolar fuzzy left* (*resp., right*) *ideal of* **S** if for any $x, y \in S$:

(1) $f(xy) \geqslant f(y)$ (resp., $f(xy) \geqslant f(x)$);
(2) $x \leqslant y$ implies $f(x) \geqslant f(y)$.

A tripolar fuzzy set of S is called a *tripolar fuzzy* (*two-sided*) *ideal* of **S** if it is a tripolar fuzzy left and a tripolar fuzzy right ideal of **S**.

Example 2.5. Let $S = \{a, b, c, d\}$. Define a binary operation $\cdot$ on S and an order relation $\leqslant$ on S as follows.

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	b	c	c
d	a	b	d	d

And $\leqslant := \Delta_S$, where $\Delta_S := \{(x, x) : x \in S\}$. Then $\mathbf{S} := \langle S; \cdot, \leqslant \rangle$ is an ordered semigroup. We define a tripolar fuzzy sets f and g of S by

x	$f^+(x)$	$f^*(x)$	$f^-(x)$
a	0.9	0.1	−1
b	0.4	0.4	−0.6
c	0.5	0.3	−0.8
d	0.5	0.3	−0.8

and

x	$g^+(x)$	$g^*(x)$	$g^-(x)$
a	0.9	0.1	−0.5
b	0.7	0.2	−0.4
c	0.1	0.6	0
d	0.2	0.8	−0.4

By careful calculation, we obtain that f is a tripolar fuzzy left ideal of $\mathbf{S}$ but not a tripolar fuzzy right ideal of $\mathbf{S}$ since there exist $b, c \in S$ such that $f^+(cb) = 0.4 < 0.5 = f^+(c)$. On the other hand, we obtain that g is a tripolar fuzzy right ideal of $\mathbf{S}$ but not a tripolar fuzzy left ideal of $\mathbf{S}$ since there exist $c, d \in S$ such that $g^+(cd) = 0.1 < 0.2 = g^+(d)$.

The illustration above demonstrates the distinction between tripolar fuzzy left ideals and tripolar right ideals in ordered semigroups.

3. Main results

This section provides a connection between (left, right) ideals and tripolar fuzzy (left, right) ideals in ordered semigroups. Later, we characterize tripolar fuzzy (left, right) ideals in ordered semigroups in terms of the operation $\circ$. Lastly, we apply such characterizations to classify particular classes of ordered semigroups.

Firstly, we show how left ideals relate to tripolar fuzzy sets in ordered semigroups.

Lemma 3.1. *Let $\mathbf{S}$ be an ordered semigroup and A a nonempty subset of S. Then the following statements are equivalent:*

(1) *A is a left ideal of $\mathbf{S}$;*
(2) *χ_A is a tripolar fuzzy left ideal of $\mathbf{S}$.*

Proof.
(1) ⇒ (2) Let $x, y \in S$. If $xy \in A$, then

$$\chi_A^+(xy) = 1 \geqslant \chi_A^+(y), \quad \chi_A^*(xy) = 0 \leqslant \chi_A^*(y), \quad \text{and} \quad \chi_A^-(xy) = -1 \leqslant \chi_A^-(y).$$

If $xy \notin A$, then $y \notin A$. Thus,

$$\chi_A^+(xy) = \chi_A^+(y), \quad \chi_A^*(xy) = \chi_A^*(y), \quad \text{and} \quad \chi_A^-(xy) = \chi_A^-(y).$$

Therefore, $\chi_A(xy) \geqslant \chi_A(y)$. Suppose that $x \leqslant y$. If $y \in A$, then $x \in A$. Thus,

$$\chi_A^+(x) = \chi_A^+(y), \quad \chi_A^*(x) = \chi_A^*(y), \quad \text{and} \quad \chi_A^-(x) = \chi_A^-(y).$$

If $y \notin A$, then

$$\chi_A^+(x) \geqslant 0 = \chi_A^+(y), \quad \chi_A^*(x) \leqslant 1 = \chi_A^*(y), \quad \text{and} \quad \chi_A^-(x) \leqslant 0 = \chi_A^-(y).$$

Therefore, $\chi_A(x) \geqslant \chi_A(y)$. Altogether, we have that χ_A is a tripolar fuzzy left ideal of $\mathbf{S}$.

(2) $\Rightarrow$ (1) Let $x \in S$ and $y \in A$. Then $\chi_A(xy) \geqslant \chi_A(y)$. By Remark 2.3, we have that $xy \in A$. Now, let $x, y \in S$ such that $x \leqslant y$ and $y \in A$. Then $\chi_A(x) \geqslant \chi_A(y)$. Again, by Remark 2.3, it follows that $x \in A$. Thus, A is a left ideal of $\mathbf{S}$. □

With some little modifications, we obtain the following result.

Lemma 3.2. *Let* $\mathbf{S}$ *be an ordered semigroup and* A *a nonempty subset of* S. *Then the following statements are equivalent:*

(1) A *is a (right) ideal of* $\mathbf{S}$*;*
(2) χ_A *is a tripolar fuzzy (right) ideal of* $\mathbf{S}$*.*

Lemmas 3.1 and 3.2 show how tripolar fuzzy left (resp., right, two-sided) ideals can be used to describe left (resp., right, two-sided) ideals in ordered semigroups. The following results illustrate the characterizations of tripolar fuzzy ideals using the operation $\circ$ on the set of all tripolar fuzzy sets of S.

Proposition 3.3. *Let* $\mathbf{S}$ *be an ordered semigroup and* f *a tripolar fuzzy set of* S. *Then the following statements are equivalent:*

(1) f *is a tripolar fuzzy left ideal of* $\mathbf{S}$*;*
(2) f *satisfies the following statements:*
   (a) $1 \circ f \subseteq f$*;*
   (b) $f(x) \geqslant f(y)$ *whenever* $x \leqslant y$ *for all* $x, y \in S$.

Proof.
(1) $\Rightarrow$ (2) It is sufficient to show only that $1 \circ f \subseteq f$. Let $a \in S$. If $\mathbf{S}_a = \emptyset$, then

$$(1 \circ f)^+(a) = 0 \leqslant f^+(a), \quad (1 \circ f)^*(a) = 1 \geqslant f^*(a), \quad \text{and} \quad (1 \circ f)^-(a) = 0 \geqslant f^-(a).$$

Suppose that $\mathbf{S}_a \neq \emptyset$. Then

$$(1 \circ f)^+(a) = \bigvee_{(u,v) \in \mathbf{S}_a} \{\min\{1^+(u), f^+(v)\}\} = \bigvee_{(u,v) \in \mathbf{S}_a} f^+(v) \leqslant \bigvee_{(u,v) \in \mathbf{S}_a} f^+(uv) \leqslant \bigvee_{(u,v) \in \mathbf{S}_a} f^+(a) = f^+(a),$$
$$(1 \circ f)^*(a) = \bigwedge_{(u,v) \in \mathbf{S}_a} \{\max\{1^*(u), f^*(v)\}\} = \bigwedge_{(u,v) \in \mathbf{S}_a} f^*(v) \geqslant \bigwedge_{(u,v) \in \mathbf{S}_a} f^*(uv) \geqslant \bigwedge_{(u,v) \in \mathbf{S}_a} f^*(a) = f^*(a),$$

and

$$(1 \circ f)^-(a) = \bigwedge_{(u,v) \in \mathbf{S}_a} \{\max\{1^-(u), f^-(v)\}\} = \bigwedge_{(u,v) \in \mathbf{S}_a} f^-(v) \geqslant \bigwedge_{(u,v) \in \mathbf{S}_a} f^-(uv) \geqslant \bigwedge_{(u,v) \in \mathbf{S}_a} f^-(a) = f^-(a).$$

This shows that $(1 \circ f)(a) \leqslant f(a)$. That is, $(1 \circ f) \subseteq f$.

(2) $\Rightarrow$ (1) We need to verify only that $f(xy) \geqslant f(y)$ for all $x, y \in S$. Let $x, y \in S$. Then

$$f^+(xy) \geqslant (1 \circ f)^+(xy) = \bigvee_{(u,v) \in \mathbf{S}_{xy}} \{\min\{1^+(u), f^+(v)\}\} \geqslant \min\{1^+(x), f^+(y)\} = f^+(y),$$
$$f^*(xy) \leqslant (1 \circ f)^*(xy) = \bigwedge_{(u,v) \in \mathbf{S}_{xy}} \{\max\{1^*(u), f^*(v)\}\} \leqslant \max\{1^*(x), f^*(y)\} = f^*(y),$$

and

$$f^-(xy) \leqslant (1 \circ f)^-(xy) = \bigwedge_{(u,v) \in \mathbf{S}_{xy}} \{\max\{1^-(u), f^-(v)\}\} \leqslant \max\{1^-(x), f^-(y)\} = f^-(y).$$

This shows that $f(xy) \geqslant f(y)$. That is, f is a tripolar fuzzy left ideal of $\mathbf{S}$. □

Similarly, we can prove the following propositions.

Proposition 3.4. *Let* $\mathbf{S}$ *be an ordered semigroup and* f *a tripolar fuzzy set of* S*. Then the following statements are equivalent:*

(1) f *is a tripolar fuzzy right ideal of* $\mathbf{S}$*;*
(2) f *satisfies the following statements:*
 (a) $f \circ 1 \subseteq f$*;*
 (b) $f(x) \geqslant f(y)$ *whenever* $x \leqslant y$ *for all* $x, y \in S$*.*

Proposition 3.5. *Let* $\mathbf{S}$ *be an ordered semigroup and* f *a tripolar fuzzy set of* S*. Then the following statements are equivalent:*

(1) f *is a tripolar fuzzy ideal of* $\mathbf{S}$*;*
(2) f *satisfies the following statements:*
 (a) $f \circ 1 \subseteq f$ *and* $1 \circ f \subseteq f$*;*
 (b) $f(x) \geqslant f(y)$ *whenever* $x \leqslant y$ *for all* $x, y \in S$*.*

Later, we apply the notion of tripolar fuzzy ideals in ordered semigroups to characterize three classes of ordered semigroups; regular, intra-regular, and both regular and intra-regular ordered semigroups. We can explain these particular classes of ordered semigroups with the help of the following results.

Lemma 3.6. *Let* $\mathbf{S}$ *be an ordered semigroup, and* $A, B \subseteq S$*. Then*

(1) $\chi_A \subseteq \chi_B$ *if and only if* $A \subseteq B$*;*
(2) $\chi_A \circ \chi_B = \chi_{(AB]}$*;*
(3) $\chi_A \cap \chi_B = \chi_{A\cap B}$*.*

Proof. The propositions (1) and (3) are not difficult to prove. Hence, we prove only (2).

(2) Let $x \in S$. It is not difficult to see that if $x \notin (AB]$, that is, there is no $a \in A$ and $b \in B$ such that $x \leqslant ab$, then

$$\chi^+_{(AB]}(x) = 0 = (\chi_A \circ \chi_B)^+(x), \quad \chi^*_{(AB]}(x) = 1 = (\chi_A \circ \chi_B)^*(x), \quad \text{and} \quad \chi^-_{(AB]}(x) = 0 = (\chi_A \circ \chi_B)^-(x).$$

Suppose that $x \in (AB]$. That is, there exist $a \in A$ and $b \in B$ such that $x \leqslant ab$. Then

$$\chi^+_{(AB]}(x) = 1 = \min\{\chi^+_A(a), \chi^+_B(b)\} = \bigvee_{(u,v)\in \mathbf{S}_x} \{\min\{\chi^+_A(u), \chi^+_B(v)\}\} = (\chi_A \circ \chi_B)^+(x),$$
$$\chi^*_{(AB]}(x) = 0 = \max\{\chi^*_A(a), \chi^*_B(b)\} = \bigwedge_{(u,v)\in \mathbf{S}_x} \{\max\{\chi^*_A(u), \chi^*_B(v)\}\} = (\chi_A \circ \chi_B)^*(x),$$

and

$$\chi^-_{(AB]}(x) = -1 = \max\{\chi^-_A(a), \chi^-_B(b)\} = \bigwedge_{(u,v)\in \mathbf{S}_x} \{\max\{\chi^-_A(u), \chi^-_B(v)\}\} = (\chi_A \circ \chi_B)^-(x).$$

Therefore, $\chi_A \circ \chi_B = \chi_{(AB]}$. □

Lemma 3.7. *Let* $\mathbf{S}$ *be an ordered semigroup. Then* $f \circ g \subseteq f \cap g$ *for any tripolar fuzzy left ideal* g *and tripolar fuzzy right ideal* f *of* $\mathbf{S}$*.*

Proof. Let g and f be a tripolar fuzzy left and a tripolar fuzzy right ideal of $\mathbf{S}$, respectively. Given $a \in S$, if $\mathbf{S}_a = \emptyset$, then $(f \circ g)(a) = 0(a) \leqslant (f \cap g)(a)$. Suppose that $\mathbf{S}_a \neq \emptyset$. We observe that $f \subseteq 1$ and $g \subseteq 1$. Then, by Propositions 3.3 and 3.4, we have

$$(f \circ g)^+(a) \leqslant (1 \circ g)^+(a) \leqslant g^+(a) \quad \text{and} \quad (f \circ g)^+(a) \leqslant (f \circ 1)^+(a) \leqslant f^+(a).$$

This implies that $(f\circ g)^{+}(a)\leqslant \min\{g^{+}(a), f^{+}(a)\}$. Similarly, by Propositions 3.3 and 3.4, we have

$$(f\circ g)^{*}(a)\geqslant (1\circ g)^{*}(a)\geqslant g^{*}(a) \quad \text{and} \quad (f\circ g)^{*}(a)\geqslant (f\circ 1)^{*}(a)\geqslant f^{*}(a).$$

This implies that $(f\circ g)^{*}(a)\geqslant \max\{g^{*}(a), f^{*}(a)\}$. Again, by Propositions 3.3 and 3.4, we have

$$(f\circ g)^{-}(a)\geqslant (1\circ g)^{-}(a)\geqslant g^{-}(a) \quad \text{and} \quad (f\circ g)^{-}(a)\geqslant (f\circ 1)^{-}(a)\geqslant f^{-}(a).$$

This implies that $(f\circ g)^{-}(a)\geqslant \max\{g^{-}(a), f^{-}(a)\}$. Thus, $(f\circ g)(a)\leqslant (f\cap g)(a)$. Altogether, we obtain our claim. □

Now, we are ready to characterize regular ordered semigroups using tripolar fuzzy left and right ideals.

Theorem 3.8. *Let* **S** *be an ordered semigroup. Then the following statements are equivalent:*

(1) **S** *is regular;*
(2) $f\circ g = f\cap g$ *for every tripolar fuzzy left ideal* g *and tripolar fuzzy right ideal* f *of* **S**.

Proof.
(1) ⇒ (2) Let g and f be a tripolar fuzzy left and a tripolar fuzzy right ideal of **S**, respectively. By Lemma 3.7, we need to illustrate only that $f\cap g\subseteq f\circ g$. Let $a\in S$. Since **S** is regular, there exists $x\in S$ such that $a\leqslant axa = a(xa)$. This means that $\mathbf{S}_a\neq\emptyset$. Then

$$(f\circ g)^{+}(a)=\bigvee_{(u,v)\in \mathbf{S}_a}\{\min\{f^{+}(u), g^{+}(v)\}\}\geqslant \min\{f^{+}(a), g^{+}(xa)\}\geqslant \min\{f^{+}(a), g^{+}(a)\},$$
$$(f\circ g)^{*}(a)=\bigwedge_{(u,v)\in \mathbf{S}_a}\{\max\{f^{*}(u), g^{*}(v)\}\}\leqslant \max\{f^{*}(a), g^{*}(xa)\}\leqslant \max\{f^{*}(a), g^{*}(a)\},$$

and

$$(f\circ g)^{-}(a)=\bigwedge_{(u,v)\in \mathbf{S}_a}\{\max\{f^{-}(u), g^{-}(v)\}\}\leqslant \max\{f^{-}(a), g^{-}(xa)\}\leqslant \max\{f^{-}(a), g^{-}(a)\}.$$

This shows that $f\cap g\subseteq f\circ g$. Thus, by Lemma 3.7, $f\circ g = f\cap g$.

(2) ⇒ (1) Let L and R be a left and a right ideal of **S**, respectively. By Lemmas 3.1 and 3.2, we have that χ_L and χ_R are a tripolar fuzzy left and a tripolar fuzzy right ideal of **S**, respectively. By our hypothesis and Lemma 3.6, we have

$$\chi_{R\cap L} = \chi_R\cap\chi_L = \chi_R\circ\chi_L = \chi_{(RL]}.$$

This implies, by Lemma 3.6, that $R\cap L = (RL]$. By Lemma 2.1, **S** is regular. □

In our next result, we describe intra-regular ordered semigroups by tripolar fuzzy left and tripolar fuzzy right ideals of **S**.

Theorem 3.9. *Let* **S** *be an ordered semigroup. Then the following statements are equivalent:*

(1) **S** *is intra-regular;*
(2) $f\cap g\subseteq f\circ g$ *for every tripolar fuzzy left ideal* f *and tripolar fuzzy right ideal* g *of* **S**.

Proof.
(1) ⇒ (2) Let f and g be a tripolar fuzzy left and a tripolar fuzzy right ideal of **S**, respectively. Let $a\in S$.

Since $\mathbf{S}$ is intra-regular, there exist $x, y \in S$ such that $a \leqslant xa^2y = (xa)(ay)$. This means that $\mathbf{S}_a \neq \emptyset$. Then

$$(f \circ g)^+(a) = \bigvee_{(u,v)\in \mathbf{S}_a} \{\min\{f^+(u), g^+(v)\}\} \geqslant \min\{f^+(xa), g^+(ay)\} \geqslant \min\{f^+(a), g^+(a)\},$$

$$(f \circ g)^*(a) = \bigwedge_{(u,v)\in \mathbf{S}_a} \{\max\{f^*(u), g^*(v)\}\} \leqslant \max\{f^*(xa), g^*(ay)\} \leqslant \max\{f^*(a), g^*(a)\},$$

and

$$(f \circ g)^-(a) = \bigwedge_{(u,v)\in \mathbf{S}_a} \{\max\{f^-(u), g^-(v)\}\} \leqslant \max\{f^-(xa), g^-(ay)\} \leqslant \max\{f^-(a), g^-(a)\}.$$

This shows that $f \cap g \subseteq f \circ g$.

(2) ⇒ (1) Let L and R be a left and a right ideal of $\mathbf{S}$, respectively. By Lemmas 3.1 and 3.2, we have that χ_L and χ_R are a tripolar fuzzy left and a tripolar fuzzy right ideal of $\mathbf{S}$, respectively. By our hypothesis and Lemma 3.6, we have

$$\chi_{L\cap R} = \chi_L \cap \chi_R \subseteq \chi_L \circ \chi_R = \chi_{(LR]}.$$

This implies, by Lemma 3.6, that $L \cap R \subseteq (LR]$. By Lemma 2.1, $\mathbf{S}$ is intra-regular. □

Our final conclusion follows from Theorems 3.8 and 3.9.

Theorem 3.10. *Let* $\mathbf{S}$ *be an ordered semigroup. Then the following statements are equivalent:*

(1) $\mathbf{S}$ *is both regular and intra-regular;*

(2) $f \cap g \subseteq (g \circ f) \cap (f \circ g)$ *for every tripolar fuzzy left ideal* f *and tripolar fuzzy right ideal* g *of* $\mathbf{S}$.

4. Conclusion

We concentrate on a generalization of bipolar and intuitionistic fuzzy left (resp., right) ideals in ordered semigroups in this study. We establish a link between tripolar fuzzy left (resp., right) ideals and left (resp., right) ideals. A binary operation defined on the set of all tripolar fuzzy sets of ordered semigroups was used to describe tripolar fuzzy left (resp., right) ideals. Lastly, we use the notion of tripolar fuzzy left (resp., right) ideals to represent regular and intra-regular ordered semigroups. As a result, Lemma 2.1 is generalized by our determinations. Since many algebraic systems can be studied using tripolar fuzzy sets, it is still unknown if the same idea can be utilized to investigate hyperalgebraic systems.

References

[1] K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **20** (1986), 87–96. 1

[2] Y. Cao, *Characterizations of regular ordered semigroups by strong quasi-ideals*, Vietnam J. Math., **30** (2002), 239–250. 2.1

[3] N. Kehayopulu, *On completely regular poe-semigroups*, Math. Japon., **37** (1992), 123–130. 2, 1

[4] N. Kehayopulu, *On intra-regular ordered semigroups*, Semigroup Forum, **46** (1993), 271–278. 2

[5] D. M. Lee, S. K. Lee, *On intra-regular ordered semigroups*, Kangweon-Kyungki Math. J., **14** (2006), 95–100. 2.1

[6] K. M. Lee, K. M. Lee, K. J. Cios, *Comparison of interval-valued fuzzy sets, intuitionistic fuzzy sets, and bipolar-valued fuzzy sets*, Comput. Inf. Technol., (2001), 433–439. 1

[7] M. M. K. Rao, *Tripolar fuzzy interior ideals of Γ-semigroup*, Ann. Fuzzy Math. Inform., **15** (2018), 199–206. 1, 2.2

[8] M. M. K. Rao, B. Venkateswarlu, *Tripolar fuzzy interior ideals of a Γ-semiring*. Asia Pac. J. Math., **5** (2018), 192–207. 1

[9] N. Wattanasiripong, N. Lekkoksung, S. Lekkoksung, *On tripolar fuzzy interior ideals in ordered semigroups*, Int. J. Innov. Comput. Inf. Control, **18** (2022), 1291–1304. 1, 2.4

[10] L. A. Zadeh, *Fuzzy sets*, Inf. Control, **3** (1965), 338–353. 1

[11] W. R. Zhang, *Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis*, In: NAFIPS/IFIS/NASA'94. Proceedings of the first international joint conference of the North American fuzzy information processing society biannual conference, The Industrial Fuzzy Control and Intellige, IEEE, (1994), 305–309. 1